Contours:-

Length of Contour:-

Suppose C be an arc having parametric representation

$$z(t) = x(t) + iy(t) \quad (a \le t \le b) \quad — (1)$$

Now suppose that the components $x'(t)$ and $y'(t)$ of (1) are continuous on the entire interval $a \le t \le b$. Then the arc C is called a <u>differentiable arc</u>, and the real-valued function

$$|z'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$$

is integrable over interval $a \le t \le b$.

Then length of arc C is

$$L = \int_a^b |z'(t)|\, dt .$$

and this length 'L' of an arc C is invariant under change of representation of C, let C be an arc as in fig 1.

Then we can change the interval over which the parameter ranges to any other interval.

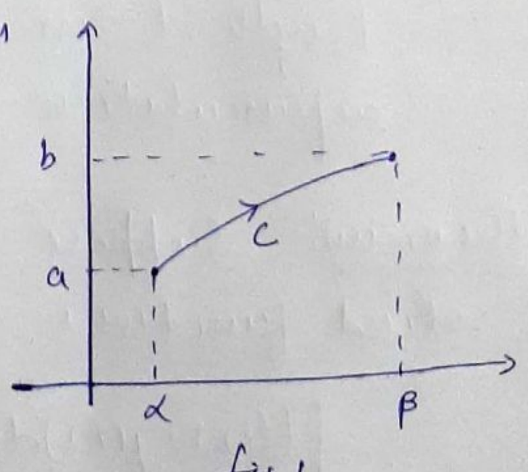


fig 1.

Suppose that
$$t = \phi(\tau) \quad (\alpha \le \tau \le \beta)$$
where ϕ is real valued function on interval $\alpha \le \tau \le \beta$ onto the interval $a \le t \le b$.

i.e. $\phi : [\alpha, \beta] \to [a, b]$

and $\phi(\alpha) = a$, $\phi(\beta) = b$.

Assume ϕ is continuous with a continuous derivative. then representation ① is transformed as
$$z = Z(\tau) \quad (\alpha \le \tau \le \beta)$$
where $Z(\tau) = z(\phi(\tau))$

then $L = \int_{\alpha}^{\beta} |Z'(\tau)| \, d\tau$

and $Z'(\tau) = z'(\phi(\tau)) \phi'(\tau)$

$$\therefore \quad L = \int_{\alpha}^{\beta} |z'(\phi(\tau))| \phi'(\tau) \, d\tau$$
$$= \int_{a}^{b} |z'(t)| \, dt \quad \left(\text{using } t = \phi(\tau) \right).$$

$\therefore$ Length of an arc is invariant under parametric representation of arc.

<u>Theorem</u>:- Suppose f and g are two complex valued functions and C be a contour. Then
$$\int_C (f(z) + g(z)) \, dz = \int_C f(z) \, dz + \int_C g(z) \, dz$$

Proof:- Suppose $z = z(t)$ $(a \le t \le b)$

represents the contour C.

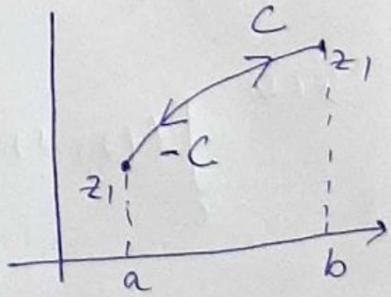
then $\cancel{\int_C f(z)\, dt}$

$$\int_C (f(z)+g(z))\,dz = \int_a^b [f(z(t))+g(z(t))]\, z'(t)\,dt$$

$$= \int_a^b f(z(t))\, z'(t)\,dt + \int_a^b g(z(t))\, z'(t)\,dt$$

$$= \int_C f(z)\,dz + \int_C g(z)\,dz .$$

Thm:- (2) $\displaystyle\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$



→ Suppose

$z = z(t)$ $(a \le t \le b)$

represents the contour C.

then

$z = z(-t)$ $(-b \le t \le -a)$

represents the contour $-C$.

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f[z(-t)]\, \frac{d}{dt} z(-t)\,.\,dt$$

$$= -\int_{-b}^{-a} f(z(-t))\, z'(-t)\,dt = -\int_a^b f(z(t))z(t)\,dt$$

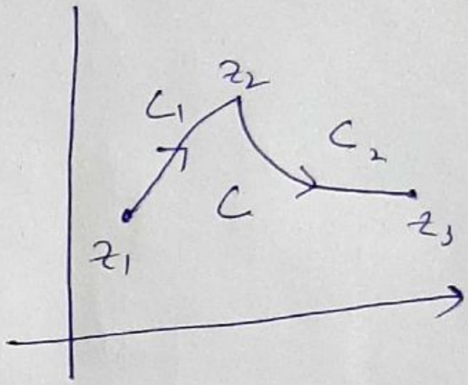
$$\therefore \int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$$

<u>Theorem:-</u> Consider a arc C, that consists of a contour C_1 and contour C_2, then

$$\int_{C} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz .$$

<u>Proof:-</u> let C_1 is represented by

$z = z(t) \quad (a \le t \le c)$



and C_2 is represented by

$z = z(t) \quad (c \le t \le b)$

and C is represented by

$z = z(t) \quad (a \le t \le b)$

then

$$\int_{C} f(z)\,dz = \int_{a}^{b} f[z(t)]z'(t)\,dt$$

$$= \int_{a}^{c} f[z(t)]z'(t)\,dt + \int_{c}^{b} f[z(t)]z'(t)\,dt$$

$$= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz .$$

Theorem (1):- If $w(t)$ is a piecewise continuous complex valued function defined on an interval $a \le t \le b$, then
$$\left| \int_a^b w(t)\,dt \right| \le \int_a^b |w(t)|\,dt .$$

Proof:- If $\int_a^b w(t)\,dt$ 0, then inequality holds trivially.

Suppose that the value of $\int_a^b w(t)\,dt$ is non zero complex number.

then $\int_a^b w(t)\,dt = r_0 e^{i\theta}$ where $r_0 = \left| \int_a^b w(t)\,dt \right|$

($\because$ $\int_a^b w(t)\,dt$ is a complex no., then have ~~parametric~~ exponential form)

$$\therefore \quad r_0 = e^{-i\theta} \int_a^b w(t)\,dt$$
$$= \int_a^b e^{-i\theta} w(t)\,dt$$

Now as r_0 is a real no., then

$\int_a^b e^{-i\theta} w(t)\,dt$ is also a real no. and

$$r = Re \int_a^b e^{-i\theta} w(t)\,dt = \int_a^b Re\left[e^{-i\theta} w(t)\right] dt$$

But

$$Re[e^{-i\theta} w(t)] \leq |e^{-i\theta} w(t)| = |e^{-i\theta}||w(t)| = |w(t)|$$

$$(\because |e^{-i\theta}| = 1).$$

$$\therefore \int_a^b Re[e^{-i\theta} w(t)]\,dt \leq \int_a^b |w(t)|\,dt$$ [being real valued integral]

$$\Rightarrow \quad r_0 \leq \int_a^b |w(t)|\,dt$$

But $\quad r_0 = \left|\int_a^b w(t)\,dt\right|$

$$\therefore \left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt.$$

(2)

Theorem:- Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a non negative constant such that $|f(z)| \leq M$. for all z on C at which $f(z)$ is defined, then

$$\left|\int_C f(z)\,dz\right| \leq ML.$$

Proof:- Suppose $z = z(t)$ $(a \leq t \leq b)$ be the parametric representation of C.

then

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f[z(t)]z'(t)\,dt\right|$$

$$\leq \int_a^b |f[z(t)]z'(t)|\,dt \quad (\text{By thm } ①)$$

$$= \int_a^b |f[z(t)]|\,|z'(t)|\,dt$$

$$\leq \int_a^b M|z'(t)|\,dt \quad \left(\because |f(z)| \leq M \text{ on } C\right)$$

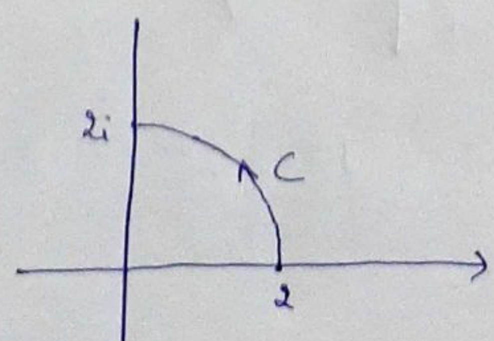
$$= M\int_a^b |z'(t)|\,dt$$

$$= ML \quad \left(\because L = \int_a^b |z'(t)|\,dt\right).$$

$$\therefore \left|\int_C f(z)\,dz\right| \leq ML$$

Example

① Let C be arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$



show that $\left| \int_C \frac{z+4}{z^3-1} dz \right| \le \frac{6\pi}{7}$

→ length of arc

$$L = \int_0^{\pi/2} |z'(t)| dt \quad , \text{where } z(t) = 2e^{it}$$

$$= \int_0^{\pi/2} |2ie^{it}| dt$$

$$= \int_0^{\pi/2} 2 \, dt = \pi .$$

$$\therefore \quad L = \pi$$

Also, $|z+4| \le |z| + 4 = 6$ on C $(\because |z| = 2)$

and

$$|z^3 - 1| \ge \left| |z^3| - 1 \right| = 7$$

$\therefore \left| \frac{z+4}{z^3-1} \right| \le \frac{6}{7}$ where z lies on C

$\therefore$ By theorem 2.

$$\left| \int_C \frac{z+4}{z^3-1} \right| \le \frac{6}{7} \pi .$$